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Maximal subgroups of $GL_n(D)$

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Abstract

In this paper we study the structure of locally solvable, solvable, locally nilpotent, and nilpotent maximal subgroups of skew linear groups. In [S. Akbari et al., J. Algebra 217 (1999) 422–433] it has been conjectured that if D is a division ring and M a nilpotent maximal subgroup of D^*, then D is commutative. In connection with this conjecture we show that if $F[M]\backslash F$ contains an algebraic element over F, then M is an abelian group. Also we show that $\mathbb{C}^* \cup \mathbb{C}^*\mathrm{j}$ is a solvable maximal subgroup of real quaternions and so give a counterexample to Conjecture 3 of [S. Akbari et al., J. Algebra 217 (1999) 422–433], which states that if D is a division ring and M a solvable maximal subgroup of D^*, then D is commutative. Also we completely determine the structure of division rings with a non-abelian algebraic locally solvable maximal subgroup, which gives a full solution to both cases given in Theorem 8 of [S. Akbari et al., J. Algebra 217 (1999) 422–433]. Ultimately, we extend our results to the general skew linear groups.

Keywords: Division rings; Skew linear groups; Nilpotent; Irreducible; Maximal subgroups

Introduction

The theory of linear groups over division rings of a fairly general type has been intensively developed in recent decades. The most important results concerning skew linear groups can be found in the interesting book of Shirvani and Wehrfritz (1986) [19].

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Since typically maximal subgroups are large, their algebraic properties carry over to those of the whole group. In this article we are interested in investigating what happens whenever a maximal subgroup of a general skew linear group admits some specific algebraic property. There are many articles devoted to investigations on skew linear groups include one-dimensional case, i.e., $GL_1(D)$, the multiplicative group of a division ring D. Recently the structure of maximal subgroups of division rings have been studied; for instance, see [1–3]. In [3] the following two conjectures have appeared.

Conjecture A. *Let D be a division ring and M a solvable maximal subgroup of D^*. Then D is commutative.*

Conjecture B. *Let D be a division ring and M a nilpotent maximal subgroup of D^*. Then D is commutative.*

The special case of Conjecture B makes the following interesting conjecture.

Conjecture C. *Let D be a division ring and M an abelian maximal subgroup of D^*. Then D is commutative.*

The present article contains three sections. In Section 1 we establish that $\mathbb{C}^* \cup \mathbb{C}^*\mathrm{j}$ ($\mathbb{C}$ is the field of complex numbers) is a solvable maximal subgroup of $\mathbb{H}^*$, where $\mathbb{H}$ denotes real quaternions. Thus this gives a counterexample to Conjecture A.

Also in Section 1 we show that if M is a nilpotent maximal subgroup of D^*, where D is a division ring and $M\backslash F$ contains an algebraic element over F, then M is an abelian group. In Section 2 we construct a family of maximal subgroups of $GL_n(D)$, $n > 1$, and completely classify the reducible maximal subgroups of $GL_n(D)$. Moreover, Conjecture C will be extended to skew linear groups and it will be shown that if D is an infinite-dimensional division ring over its center F and $GL_n(D)$ is algebraic over F, then $GL_n(D)$ contains no abelian maximal subgroup. Finally, in Section 3 we generalize our results to general skew linear groups and prove that for a locally finite-dimensional division ring D, if M is a locally nilpotent maximal subgroup of $GL_n(D)$, then M is abelian. Also we show that locally finiteness of a maximal subgroup of $GL_n(D)$ implies that $GL_n(D)$ is locally finite.

Before stating our results, we fix some notations. Throughout, D is a division ring with center F, we shall denote their multiplicative groups by D^*, F^*, respectively. For any group G and any subset X of G, $C_G(X)$, and $N_G(X)$ denote respectively the centralizer and the normalizer of X in G. In this article G^r denotes the rth derived subgroup of G. The set of primes which occur as the order of some element of G will be denoted by $\Pi(G)$. Throughout this paper $H < G$ means that H is a not necessarily proper subgroup of G. By a skew linear group we mean a subgroup of the general skew linear group $GL_n(D)$, for some division ring D. Let G be a subgroup of $GL_n(D)$ and set $V = D^n$, the space of row n-vectors over D. Then V is a D-G bimodule in the obvious way. We say that G is an irreducible (respectively reducible, completely reducible) subgroup of $GL_n(D)$, whenever V is irreducible (respectively reducible, completely reducible) as a D-G bimodule. Also G is said to be absolutely irreducible if $F[G] = M_n(D)$. For a field F, we call a subgroup

G of $GL_n(F)$ nonmodular if G is torsion and $\mathrm{Char}\, F \notin \Pi(G)$. A field extension K/F is said to be radical if for every $x \in K$ there exists a natural number $n(x)$ such that $x^{n(x)} \in F$. For a subset X of $M_n(D)$ and a subring R of D, $R[X]$ denotes the subalgebra generated by R and X. Also if X is a subset of D, then $R(X)$ denotes the subdivision ring generated by R and X. We write $[V:K]_r$ for the dimension of a right vector space V over a field K. For a given ring R, the group of units of R is denoted by $U(R)$.

1. Maximal subgroups of division rings

Division rings have a more complicated structure than fields and much less is known about them. It is easily seen that the multiplicative group of a field does not necessarily have a maximal subgroup. For example, if M is a maximal subgroup of index n of $\mathbb{C}^*$, then for any $x \in \mathbb{C}^*$, x^n is contained in M, implying $M = \mathbb{C}^*$, since indeed any element of $\mathbb{C}^*$ is an nth power. It seems that the multiplicative group of a noncommutative division ring has maximal subgroup and it was conjectured first in [2]. The structure of a maximal subgroup of a group forces some properties in the group. For example, in [17] there is a theorem which says that if a finite group contains an abelian maximal subgroup, then it should be solvable. It is not true in general case, because there are some examples of nonsolvable groups which have finite abelian maximal subgroups, see [15]. Since $GL_n(D)$ is violently non-abelian it is intuitively see that $GL_n(D)$ has no abelian maximal subgroup. It is a natural question to ask what can be said about the structure of division rings whose multiplicative subgroups have an abelian maximal group or a solvable maximal subgroup.

In what follows we shall give a counterexample to Conjecture A. Let us record the following technical lemma.

Lemma 1. $\mathbb{H}^* = \langle \mathbb{C}^*, \mathrm{j}+1 \rangle$.

Proof. Let $G = \langle \mathbb{C}^*, \mathrm{j}+1 \rangle$. We claim that if $w, z \in \mathbb{C}$ with $|z| = |w|$ and $z + \mathrm{j} \in G$ then $w + \mathrm{j} \in G$. For $z = 0$ it is clear, so assume that $z \neq 0$. Since $|z| = |w|$, there exists some element $t \in \mathbb{C}$ such that $wz^{-1} = t/\bar{t}$. Now, $(z+\mathrm{j})t = \bar{t}(w+\mathrm{j}) \in G$. So, $w + \mathrm{j} \in G$ which proves the claim.

Since each element of $\mathbb{H}$ is of the form $z_1 + z_2\mathrm{j}$ where $z_1, z_2 \in \mathbb{C}$, to prove the lemma it suffices to show that for any $a \in \mathbb{C}$ we have $a + \mathrm{j} \in G$. For $r > 0$, let

$$u = \frac{1-r}{1+r} + \left(\sqrt{1 - \left(\frac{1-r}{1+r}\right)^2}\right)\mathrm{i}.$$

So $|u| = 1$ and using our claim, since $1+\mathrm{j} \in G$, we have $u+\mathrm{j} \in G$. Thus $(1+\mathrm{j})(u+\mathrm{j}) \in G$, which implies that $(u-1)/(\bar{u}+1) + \mathrm{j} \in G$. On the other hand, we have

$$\left|\frac{u-1}{\bar{u}+1}\right| = \sqrt{r}.$$

Since $r > 0$ was arbitrary, by our claim for all $a \in \mathbb{C}$ we have $a + \mathrm{j} \in G$. $\square$

The following theorem asserts a very interesting property of real quaternions, that there is just one subgroup between $\mathbb{C}^*$ and $\mathbb{H}^*$; actually it gives a counterexample to Conjecture 3 of [3].

Theorem 1. *Let M be a proper subgroup of $\mathbb{H}^*$ containing $\mathbb{C}^*$ then $M = \mathbb{C}^*$ or $M = \mathbb{C}^* \cup \mathbb{C}^*\mathrm{j}$. Therefore $\mathbb{H}^*$ has a solvable maximal subgroup.*

Proof. It suffices to prove that for any $z_0 \in \mathbb{C}^*$, $\mathbb{H}^* = \langle \mathbb{C}^*, z_0 + \mathrm{j}\rangle$. Let $G = \langle \mathbb{C}^*, z_0 + \mathrm{j}\rangle$ where $|z_0|^2 = r_0 > 0$. By a similar method used in the proof of the previous lemma, since $z_0 + \mathrm{j} \in G$, both $\sqrt{r_0}\,\mathrm{i} + \mathrm{j}$ and $\sqrt{r_0} + \mathrm{j}$ are in G. So,

$$\big(\sqrt{r_0} + \mathrm{j}\big)\big(\sqrt{r_0}\,\mathrm{i} + \mathrm{j}\big) = (r_0\mathrm{i} - 1) + \big(\sqrt{r_0} - \sqrt{r_0}\,\mathrm{i}\big)\mathrm{j} \in G.$$

Let us put

$$z_1 = \frac{r_0\mathrm{i} - 1}{\sqrt{r_0} - \sqrt{r_0}\,\mathrm{i}}.$$

Then $|z_1|^2 = ({r_0}^2 + 1)/(2r_0) \geqslant 1$ and $z_1 + \mathrm{j} \in G$. Again let $|z_1|^2 = r_1 \geqslant 1$. Similarly $(\sqrt{r_1} + \mathrm{j})(\sqrt{r_1}\,\mathrm{i} + \mathrm{j}) \in G$. Now, if we define

$$z_2 = \frac{r_1\mathrm{i} - 1}{\sqrt{r_1} - \sqrt{r_1}\,\mathrm{i}}$$

then

$$z_2 + \mathrm{j} \in G \quad \text{and} \quad |z_2|^2 = \frac{{r_1}^2 + 1}{2r_1} \geqslant 1.$$

Continuing this method we obtain the sequences $\{r_n\}$ and $\{z_n\}$ with the following properties:

$$r_n = \frac{r_{n-1}^2 + 1}{2r_{n-1}} \geqslant 1, \qquad z_n = \frac{r_{n-1}\mathrm{i} - 1}{\sqrt{r_{n-1}} - \sqrt{r_{n-1}}\,\mathrm{i}}, \quad z_n + \mathrm{j} \in G, \quad \text{and} \quad |z_n|^2 = r_n.$$

Since the sequence $\{r_n\}$ is decreasing and bounded by 1, so when n tends to infinity r_n converges to 1. Consequently, there exists a natural number m such that $z_m + \mathrm{j} \in G$ and $(t^2 - 1)^2/(4t) < t$ where $|z_m| = t$. Now let

$$u = \frac{(t^2 - 1)^2}{4t} + \sqrt{t^2 - \left(\frac{(t^2 - 1)^2}{4t}\right)^2}\,\mathrm{i}.$$

Thus, $(t + \mathrm{j})(u + \mathrm{j}) \in G$ which implies that

$$\frac{tu - 1}{t + \bar{u}} + \mathrm{j} \in G \quad \text{and} \quad \left|\frac{tu - 1}{t + \bar{u}}\right| = 1.$$

So, $1+\mathrm{j} \in G$ and by Lemma 1 we have $G = \mathbb{H}^*$. On the other hand, $\mathbb{C}^*$ is an abelian normal subgroup of index 2 of the group $\mathbb{C}^* \cup \mathbb{C}^*\mathrm{j}$; so it is solvable. The proof is complete. □

We note the following simple fact.

Lemma 2. *Let G be a group and M a maximal subgroup of G, then either $Z(G) \subseteq M$ or $G' \subseteq M$.*

Proof. Assume that $Z(G) \nsubseteq M$, so by maximality of M we have $Z(G)M = G$. Therefore $G' = M' \subseteq M$, which completes the proof. □

Lemma 3. *Let G be a group, such that $G/Z(G)$ is a locally finite group, then G' is locally finite.*

Proof. It is sufficient to prove that for any finitely generated subgroup of G, say N, N' is finite. By the assumption $Z(G)N/Z(G)$ is finite and so is $N/N \cap Z(G)$. Therefore $N/Z(N)$ is finite, thus N' is finite. □

Lemma 4. *Let D be a division ring with at least 4 elements, and M a solvable maximal subgroup of $GL_n(D)$. If F is the center of D, then $F^* \subseteq M$ unless when $n = 1$ and $D = F$. In addition, if $n = 1$ then either M is abelian or $F(M) = D$, and so $Z(M) = F^*$.*

Proof. By Lemma 2 we have $F^* \subseteq M$ or $SL_n(D) \subseteq M$; in the later case we conclude that $SL_n(D)$ is solvable, but by [10, p. 138], we know that for $n > 1$, $(SL_n(D))' = SL_n(D)$ which is a contradiction. So $F^* \subseteq M$. If $n = 1$ and F^* is not contained in M, then by Lemma 2, $D' < M$ and so D^* is solvable and by Hua's Theorem we have $D = F$. To prove the second part of lemma, we note that since $n = 1$, we have $M \subseteq F(M)$, where $F(M)$ is a division ring. If $M = (F(M))^*$, then by Hua's Theorem [12], M is abelian. If $M \neq (F(M))^*$, then by maximality of M we obtain $F(M) = D$. □

Lemma 5. *Let D be a division ring with center F and M a nilpotent maximal subgroup of D^*, then M is a metabelian group.*

Proof. If M is abelian then M is a metabelian group. If not by Lemma 4 we have $Z(M) = F^*$. Since by assumption M/F^* is a nilpotent group then we have $Z(M/F^*)$ is nontrivial. Suppose that $F^*y \in Z(M/F^*)$ and $y \notin F$. By considering the homomorphism $\theta : M \to F^*$, taken by the rule $\theta(x) = xyx^{-1}y^{-1}$, we obtain that $M/C_M(y) < F^*$ and so $M' < C_M(y)$. If $F(M') = D$, then we should have $y \in F$, which is a contradiction. Therefore $F(M') \neq D$. On the other hand, we have $M < N_{D^*}((F(M'))^*)$ and $(F(M'))^* < N_{D^*}((F(M'))^*)$. By Cartan–Brauer–Hua Theorem [13, p. 222] we obtain $M' \subseteq F$ or $(F(M'))^* < M$, and thus by Hua's Theorem we conclude that M' is an abelian group. □

In connection with Conjecture B appeared in the introduction, we obtain the following result.

Theorem 2. *Let D be a noncommutative division ring with center F and M a nilpotent maximal subgroup of D^*; then there exists a maximal subfield of D, say K, such that $K^* \lhd M$. Furthermore M/K^* is an abelian group and K^* is indeed the unique maximal abelian normal subgroup of M.*

Proof. By Lemma 4 we have $F^* < M$. Let us put

$$\Sigma = \{N \mid N \lhd M,\ N \text{ is an abelian subgroup of } M\}.$$

We note that according to Lemma 5, $M' \in \Sigma$. By Zorn's Lemma, it is easily seen that Σ has a maximal element, say K^*. We claim that $K^* \cup \{0\} = K$ is a maximal subfield of D. Suppose that $F(K) \subseteq D$ is the field generated by F and K. We note that $M < N_{D^*}((F(K))^*)$ and so we obtain that $N_{D^*}((F(K))^*) = M$, since otherwise $N_{D^*}((F(K))^*) = D^*$ and, by Cartan–Brauer–Hua Theorem, we find $D = F(K)$ or $F(K) \subseteq F$. The first case implies that D is commutative. The second case shows that F^* is a maximal abelian normal subgroup of M and so $M' \subseteq F$. Let $x \in M\backslash F$ then we have $M < N_{D^*}((F(x))^*)$. On the other hand, $(F(x))^* < N_{D^*}((F(x))^*)$ and this implies that $(F(x))^* \lhd M$, which is a contradiction.

Thus we have, $K^* < (F(K))^* \lhd M$, and by maximality of K^* we find that $K = F(K)$. Let L be a maximal subfield of D such that $K \subseteq L$, since $N_{D^*}(K^*) = M$ we have $K^* < L^* < M$. Since $K^* \lhd M$, we have $L^* < C_M(K^*) \lhd M$. Now, if $C_M(K^*)/K^*$ is nontrivial, then noting that $C_M(K^*)/K^* \lhd M/K^*$ and M/K^* is nilpotent, we obtain that $C_M(K^*)/K^* \cap Z(M/K^*)$ is nontrivial. Thus there is an element $a \in C_M(K^*)\backslash K^*$ such that $K^*a \in Z(M/K^*)$. Now we obtain $K^* \subsetneqq \langle K^*, a\rangle \lhd M$, which is a contradiction. Therefore we have $C_M(K^*) = K^*$ which yields that $L^* = K^*$ and thus K is a maximal subfield of D.

Now suppose that K_1^* and K_2^* are two maximal abelian normal subgroups of M. As we saw before, $K_1 = K_1^* \cup \{0\}$ and $K_2 = K_2^* \cup \{0\}$ are maximal subfields of D. Choose two elements $x \in K_1^*$ and $y \in K_2^*$ such that $xyx^{-1}y^{-1} \neq 1$. We have $xy = ayx$ and $(x+1)y = by(x+1)$, where $a, b \in K_2$. This shows that $y = (b-a)yx + by$. If $b = a$, then $b = a = 1$, which is a contradiction. Therefore $x \in K_2$, which contradicts our assumption and thus Σ has a unique maximal element. Now, since $M' \in \Sigma$, M' is contained in the unique maximal element of Σ, which shows that M/K^* is an abelian group. □

Theorem 3. *If D is a division ring with center F and M a nilpotent maximal subgroup of D^*, then we have that $F[M]$ is a J-semisimple ring.*

Proof. If $F[M] = D$, the result is obvious, therefore we may assume that $U(F[M]) = M$. If J is the Jacobson radical of $F[M]$, then, since every element of $1 + J$ is unit, we conclude that $1 + J < M$. By Theorem 2, there exists a maximal subfield K of D such that M/K^* is an abelian group; thus $K^*(1+J)/K^* \simeq (1+J)/(K^* \cap (1+J))$ is an abelian group. We claim that $K^* \cap (1+J) = \{1\}$. Assume that $0 \neq a \in J$ and $1 + a \in K$; so $a \in K$. Therefore $a^{-1} \in K$, but J is an ideal of $F[M]$, so $1 \in J$, which is a contradiction. Thus we obtain that $1 + J$ is abelian and since $1 + J$ is a normal subgroup of M, by Theorem 2, $1 + J \subseteq K^*$. But we note that $K^* \cap (1+J) = \{1\}$, and this implies that $J = 0$. □

Lemma 6. *Let D be a division ring with center F.*

(i) *If K is a subfield of $M_n(D)$ such that $[M_n(D):K]_r$ is finite, then D is finite-dimensional over F.*
(ii) *If $K^* < M$, where M is a non-abelian maximal subgroup of D^* and K is a subfield of D such that $[M:K^*]$ is finite, then D is finite-dimensional over F.*

Proof. (i) Let us put $[M_n(D):K]_r = m$. It is not hard to see that $M_n(D)$ can be embedded in the matrix ring $M_{mn}(K)$. But $M_{mn}(K)$ satisfies the standard polynomial identity and so D satisfies the standard polynomial identity; thus by a theorem of [16, p. 98], D is finite-dimensional over F.

(ii) Consider the ring $F[M]$. If $U(F[M]) \neq M$ then we find $F[M] = D$ and, since $M = \bigcup_{i=1}^{n} a_i K^*$, we have $[D:K]_r < \infty$, and by part (i), the proof is complete.

Thus we may assume that $U(F[M]) = M$. We claim $M \cup \{0\}$ is a division ring. To prove this it is enough to show that for any $a, b \in M$ with $a \neq b$, $a - b \in M$. Suppose that $1 \neq x \in M$; since $[M:K^*] = l < \infty$, we have $x^t \in K$ for some natural t. If $x^t = 1$, then $x - 1$ is algebraic over F and so $(x-1)^{-1} \in F[M]$; this yields that $x - 1 \in U(F[M]) = M$. If $x^t \neq 1$ then $(x-1)(x^{t-1} + \cdots + 1) = x^t - 1 \in K^*$; this implies that $x - 1 \in U(F[M]) = M$. Put $x = ab^{-1}$; this yields that $ab^{-1} - 1 \in M$ and so $a - b \in M$. Now $[M:K^*]$ is finite and, by a result of Faith in [11], we have M is abelian, which is a contradiction. □

Lemma 7. *Let D be a division ring with center F and M a nilpotent maximal subgroup of D^* containing F^*. If $Z(M/F^*) = Z_2(M)/F^*$, then $Z_2(M)$ is abelian and $F^* \subsetneqq Z_2(M)$.*

Proof. Since M/F^* is nilpotent, it is clear that $F^* \subsetneqq Z_2(M)$. Let $D_1 = F(Z_2(M))$ be the subdivision ring generated by F and $Z_2(M)$. We have $M < N_{D^*}(D_1^*)$ and $D_1^* < N_{D^*}(D_1^*)$. If D_1^* is not contained in M, then by maximality of M and Cartan–Brauer–Hua Theorem we have $D_1 = D$. But as we saw in the proof of Lemma 5, $M' < C_M(Z_2(M))$ and so $M' < F$. Now, since $Z(M) = F$, there is an element $y \in Z_2(M) \backslash F$ and $x \in M$ such that $xy \neq yx$. Consider the following equality:

$$xyx^{-1}y^{-1} - 1 = \big(xyx^{-1}y^{-1} - x(y+1)x^{-1}(y+1)^{-1}\big)(y+1).$$

Since $M' < F$, we have $M < N_{D^*}((F(y))^*)$ and so $(F(y))^* < M$. Now, $M' < F$ implies that $xyx^{-1}y^{-1}$ and $x(y+1)x^{-1}(y+1)^{-1}$ belong to F; hence $y \in F$, a contradiction. Thus $D_1^* < M$, and so via Hua's Theorem D_1 is commutative. That is, $Z_2(M)$ is an abelian group and the proof is complete. □

Now, we are in a position to prove Conjecture B in the case $M \backslash F$ contains an algebraic element and M is not abelian.

Theorem 4. *Let D be a division ring with center F and M a nilpotent maximal subgroup of D^*. If $M \backslash F$ contains an algebraic element over F, then M is an abelian group.*

Proof. By Theorem 2, there exists a maximal subfield K of D such that M/K^* is an abelian group. Let $x \in M \backslash F$ be algebraic over F. Two cases can be considered.

Case 1 ($x \in K$). Let $f(y) \in F[y]$ be the minimal polynomial of x over F. Since every conjugate of x with respect to M is in K and as well is a root of $f(y)$, thus we have $n = [M : C_M(x)] < \infty$. Thus there is a normal subgroup L of M such that $L \subseteq C_M(x)$, and $[M : L] < \infty$. Clearly, $M < N_{D^*}((F(L))^*)$. If $F(L) = D$, then $x \in F$, a contradiction. Therefore by Cartan–Brauer–Hua Theorem and maximality of M, we obtain $L \subseteq F$ or $(F(L))^* < M$. If $L \subseteq F$ then M/F^* is finite, which contradicts Corollary 4 of [3]. In the second case, according to Hua's Theorem, $F(L)$ is commutative. Finally, since $[M : (F(L))^*] < \infty$, by Lemma 6 we conclude that $[D : F] < \infty$, and so by [3, Theorem 7], M is abelian.

Case 2 ($x \in M \backslash K$). Consider an element $y \in Z_2(M) \backslash F$; according to Lemma 7 and Theorem 2 we have $Z_2(M) \subseteq K$. On the other hand, $x^{-1}yxy^{-1} = r \in F$. If $f(t) = \sum_{i=0}^{n} a_i t^i \in F[t]$ is the minimal polynomial of x over F, then $\sum_{i=0}^{n} a_i (yxy^{-1})^i = 0$.

Now, since $yxy^{-1} = rx$, we conclude that $\sum_{i=0}^{n} a_i r^i x^i = 0$. This yields that for some $k \geqslant 1$ we should have $r^k = 1$. We have $x^{-1}yx = ry$ and so $x^{-1}y^k x = y^k$; hence $x \in C_D(y^k)$. If y is algebraic over F, by Case 1, M is an abelian group. Thus we may assume that y is not algebraic over F. Now since $y^k \in Z_2(M)$; we have $M < N_{D^*}((F(y^k))^*)$ and $(C_D(y^k))^* < N_{D^*}((F(y^k))^*)$, and so $(C_D(y^k))^* < M$. By Hua's Theorem, $C_D(y^k)$ is commutative, and so $C_D(y^k) = K$. Thus $x \in K$, a contradiction. □

In Corollary 6 we will show that the above result is valid if one replaces $M \backslash F$ by $F[M] \backslash F$ in assumption. The following theorem has been proved.

Theorem A [3]. *Let D be a noncommutative division ring with center F and M be a solvable maximal subgroup of D^*, algebraic over F. Then $F^* \subseteq M$. Also, either M/F^* is locally finite or there exists a maximal subfield K of D such that K^* is normal in M and M/K^* is locally finite. Furthermore, if $[D : F] < \infty$, then either M is the multiplicative group of a maximal subfield of D or there is a maximal subfield K of D such that K^* is normal in M, K/F is Galois and $[M : K^*] < \infty$.*

The next theorem gives a complete description of a finite-dimensional division ring with a non-abelian solvable maximal subgroup.

Theorem 5. *Let D be a finite-dimensional division ring with center F and M be a non-abelian solvable maximal subgroup of D^*. Then $[D : F] = p^2$, where p is a prime number and there exists a maximal subfield K of D such that $K^* \lhd M$, K/F is a cyclic extension, $[K : F] = p$, and the groups* Gal(K/F) *and M/K^* are isomorphic to $\mathbb{Z}_p$. Furthermore, for any $x \in M \backslash K^*$ we have $x^p \in F$, $M = \bigcup_{i=1}^{p} K^* x^i$, and $D = \bigoplus_{i=1}^{p} K x^i$.*

Proof. By Theorem A, there exists a maximal subfield K of D such that K^* is a normal subgroup of M, K/F is a Galois extension and M/K^* is a finite group. Consequently, the map $\phi : M \to \mathrm{Gal}(K/F)$, $\phi(x) = \sigma_x$, where $\sigma_x(a) = xax^{-1}$ is well defined. Since M is maximal, we have $N_{D^*}(K^*) = M$. Therefore, by Skolem–Noether Theorem we conclude that ϕ is surjective. Now we have $C_D(K) = K$ and hence $\ker\phi = K^*$, which implies that M/K^* is isomorphic to $\mathrm{Gal}(K/F)$. Now, since M is solvable, $\mathrm{Gal}(K/F)$ is also solvable. Thus, if $|\mathrm{Gal}(K/F)|$ is not prime then $\mathrm{Gal}(K/F)$ has a nontrivial normal subgroup. Let E be the subfield of K which is the fixed field of this subgroup. So we have three distinct fields $F \subset E \subset K$. By a theorem of [14, p. 36] we know that for every $\sigma \in \mathrm{Gal}(K/F)$, $\sigma(E) = E$, which implies that $M < N_{D^*}(E^*)$. Since F and E are distinct, we have $M = N_{D^*}(E^*)$. But $C_{D^*}(E^*)$ is contained in $N_{D^*}(E^*)$ which is solvable, thus it is the multiplicative group of a field containing K. So, $C_D(E) = K$. Now using Centralizer Theorem [10, p. 42] we have $E = K$, which is a contradiction. Hence, $\mathrm{Gal}(K/F)$ is of prime order, which proves the first part of the theorem.

To prove the second part we note that since $[M : K^*] = p$, for any $x \in M \backslash K^*$, we have $M = \bigcup_{i=1}^{p} K^* x^i$. Now, since D is algebraic over F, we have $F[M] = \sum_{i=1}^{p} K x^i$ is a division ring. But we know that $M < (F[M])^*$, and this implies that $F[M] = D = \sum_{i=1}^{p} K x^i$. Thus the equality $[D : K] = [K : F] = p$ implies that $D = \bigoplus_{i=1}^{p} K x^i$. Now, let $y \in M \backslash K$. Since $y^p \in K$, we have $K \subseteq C_D(y^p)$. On the other hand, $y \in C_D(y^p)$ and, by the fact that $[D : K] = p$ and p is a prime, we conclude that $C_D(y^p) = D$, that is $y^p \in F$. $\square$

Theorem B [19, p. 74]. *Let D be a division ring of characteristic zero, n a natural number, and G a locally finite subgroup of $GL_n(D)$. If G is locally solvable, then G has a metabelian normal subgroup of finite index. In particular, G is solvable.*

Theorem C [21, Corollary 1.5] or [22]. *Let G be an absolutely irreducible skew linear group of degree* 1. *If G is locally solvable, then G is abelian by locally finite.*

Theorem D [19, p. 73]. *Let D be a division ring of characteristic zero, n a natural number, and G a locally finite subgroup of $GL_n(D)$. If G is locally nilpotent, then G is isomorphic to a subgroup of $GL_{2n}(\mathbb{C})$. In particular, G has an abelian normal subgroup of rank at most n and index dividing $(2n)!$.*

In what follows we give a perfect description of the structure of division rings with an algebraic non-abelian locally solvable maximal subgroup and fully verify the both cases occurred in Theorem A.

Theorem 6. *Let D be a division ring with center F and M a non-abelian locally solvable maximal subgroup of D^*, algebraic over F. Then $[D : F] = p^2$, where p is a prime number, and there exists a maximal subfield K of D such that K/F is a cyclic extension, $[K : F] = p$, and the groups $\mathrm{Gal}(K/F)$ and M/K^* are isomorphic to $\mathbb{Z}_p$. Furthermore, for any $x \in M \backslash K^*$ we have $x^p \in F$, $M = \bigcup_{i=1}^{p} K^* x^i$, and $D = \bigoplus_{i=1}^{p} K x^i$.*

Proof. First we claim that $F[M] = D$. By contradiction suppose that $F[M] \neq D$. Thus by maximality of M we conclude that $U(F[M]) = M$. Let $a, b \in M$ and $a \neq b$. Therefore we have ab^{-1} is algebraic over F and so $ab^{-1} - 1 \in U(F[M])$. Thus we find that $ab^{-1} - 1 \in M$, therefore $a - b \in M$. So we obtain that $M \cup \{0\}$ is a division ring and, since M is absolutely irreducible in $F[M] = M \cup \{0\}$, by Theorem C there exists an abelian normal subgroup of M, say N, such that M/N is locally finite. By [17, p. 440], N is central and thus M is radical over its center. Now, by Kaplansky's Theorem [13, p. 258], M is abelian, a contradiction.

Thus we may assume that M is absolutely irreducible in D. Now by Theorem C, there exists a normal abelian subgroup N of M such that M/N is locally finite. But we note that $M < N_{D^*}((F(N))^*)$ and, by maximality of M, two cases can be considered.

Case 1 ($F(N) \subseteq F$). In this case we find that M/F^* is a locally finite group. By Lemma 3, M' is also a locally finite group. Now if $\mathrm{Char}\, F = p > 0$, then for every two elements $a, b \in M'$, $\langle a, b\rangle$ is a finite group and since characteristic of F is nonzero, this group should be cyclic and so M' is an abelian group. If $M' \subseteq F$, then M is nilpotent and by Theorem 4 we are done. So we can assume that $M' \nsubseteq F$. On the other hand, we have $M < N_{D^*}((F(M'))^*)$ and, by maximality of M, we find that $M = N_{D^*}((F(M'))^*)$. This yields $(F(M'))^* < M$. Therefore $F(M')$ is radical over F and, by Kaplansky's Lemma, $F(M')/F$ is a purely inseparable extension or F is algebraic over its prime subfield. But we note that M' is torsion and so the first case does not occur. If the second case occurs, then since M/F^* and F^* are locally finite groups by [9, p. 154], we conclude that M is locally finite. Now, since $\mathrm{Char}\, F > 0$, M is an abelian group, a contradiction.

Thus we can assume that $\mathrm{Char}\, F = 0$. We know that M' is locally finite, and locally solvable, and, by Theorem B, M is solvable. Now consider a natural number r such that $M^{r-1} \nsubseteq F$ and $M^r \subseteq F$. Consider the subdivision ring $F(M^{r-1})$ of D. If $F(M^{r-1}) \neq D$, then noting that $M < N_{D^*}((F(M^{r-1}))^*)$ and $(F(M^{r-1}))^* < N_{D^*}((F(M^{r-1}))^*)$, in the view of Cartan–Brauer–Hua Theorem, we obtain $(F(M^{r-1}))^* < M$, and by Hua's Theorem, $F(M^{r-1})$ is a field. Since M/F^* is locally finite and $(F(M^{r-1}))^* < M$, we obtain that $F(M^{r-1})$ is radical over F. Since $M^{r-1} \nsubseteq F$, then F is a proper subfield of $F(M^{r-1})$ and, by Kaplansky's Lemma, $\mathrm{Char}\, F = p > 0$, which is a contradiction. Thus we may assume that $F(M^{r-1}) = D$. Since M^{r-1} is locally finite, $F[M^{r-1}]$ is locally finite-dimensional over F. To see this, let $x_1, \ldots, x_k \in M^{r-1}$; then $G = \langle x_1, \ldots, x_k\rangle$ is finite, so $F[G]$ is finite-dimensional over F. Therefore $F(M^{r-1}) = F[M^{r-1}]$. Now we have that M^{r-1} is locally finite and $M^r \subseteq Z(M^{r-1})$, so we have that M^{r-1} is a nilpotent group. By Theorem D, one may see that there exists an abelian normal subgroup L of M^{r-1} such that $n = [M^{r-1} : L] < \infty$. Let us consider $M^{r-1} = \bigcup_{i=1}^{n} a_i L$. If $K = F(L)$, then since $D = a_1 K + \cdots + a_n K$, we have $[D : K]_r < \infty$ and, by Lemma 6, we have $[D : F] < \infty$. Now, Theorem 5 completes the proof.

Case 2 ($F(N)^* \nsubseteq F$). In this case we have $N_{D^*}((F(N))^*) = M$, thus we obtain that $(F(N))^* \subseteq M$. Now let $K = F(N)$, then we have $K^* \lhd M$ and that M/K^* is locally finite. Suppose that $a \in K \backslash F$. Then for any element $b \in M$ we have $bab^{-1} \in K$. If $f(x) \in F[x]$ is the minimal polynomial of a over F, then for any $b \in M$, bab^{-1} is also a root of $f(x)$. This implies that the number of conjugates of a with respect to M, $[M : C_M(a)]$, is finite. Thus there is a normal subgroup N_1 of M such that $N_1 < C_M(a)$ and $[M : N_1]$ is finite. We have $M < N_{D^*}((F(N_1))^*)$, and hence we conclude that $N_{D^*}((F(N_1))^*) = M$ or $N_{D^*}((F(N_1))^*) = D^*$. If $N_{D^*}((F(N_1))^*) = D^*$, then $F(N_1) = D$ or $F(N_1) \subseteq F$. But since $a \notin F$, we have $F(N_1) \neq D$. If $F(N_1) \subseteq F$, then M/F^* is finite; that contradicts Corollary 4 of [3]. Therefore we can suppose that $N_{D^*}((F(N_1))^*) = M$. In this case we find that $(F(N_1))^* < M$ and, by Hua's Theorem, that N_1 is abelian and $[M : (F(N_1))^*] < \infty$; so, by Lemma 6, we have $[D : F] < \infty$ and, by Theorem 5, the proof is complete. □

2. Maximal subgroups of general skew linear groups

The theory of skew linear groups has achieved a high degree of perfection in recent years. In this section we shall touch on problems concerning maximal subgroups of general skew linear groups. First, we shall give a family of maximal subgroups of $GL_n(D)$, where D is a division ring and $n > 1$. The main questions here are those of describing properties of maximal subgroups of $GL_n(D)$, where D is a division ring. In [10, p. 140] it has been proved that, for a division ring D and two natural numbers $0 < r < n$, the set of all elements $[a_{ij}] \in GL_n(D)$ such that $a_{rj} = 0$ for $j \neq r$ is a maximal subgroup of $GL_n(D)$. Now we extend this result in the following theorem.

Theorem 7. *Let D be a division ring and r, n natural numbers such that $0 < r < n$, then the group*

$$M = \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \,\middle|\, A \in GL_r(D),\ C \in GL_{n-r}(D),\ B \in M_{r\times(n-r)}(D) \right\}$$

is a maximal subgroup of $GL_n(D)$.

Proof. Suppose that

$$x = \begin{bmatrix} A & B \\ E & C \end{bmatrix} \in GL_n(D) \setminus M,$$

so $E \neq 0$; put $G = \langle M, x \rangle$. It suffices to prove that $G = GL_n(D)$. First we claim that

$$\text{if} \quad S = \begin{bmatrix} X & Y \\ T & Z \end{bmatrix} \in G, \quad \text{then} \quad \begin{bmatrix} I_r & 0 \\ T & I_{n-r} \end{bmatrix} \in G.$$

Since S is invertible, the first r columns of S are linearly independent over D (from the right side). Thus there are r rows of the matrix $\left[\begin{smallmatrix} X \\ T \end{smallmatrix}\right]$ which are linearly independent over D (from the left side). Thus noting to elementary matrices in M we conclude that $\left[\begin{smallmatrix} I_r & Y_1 \\ T & Z_1 \end{smallmatrix}\right] \in G$. So the matrix

$$\begin{bmatrix} I_r & Y_1 \\ T & Z_1 \end{bmatrix} \begin{bmatrix} I_r & -Y_1 \\ 0 & I_{n-r} \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ T & Z_2 \end{bmatrix}$$

is in G. Therefore, Z_2 is invertible. Thus,

$$\begin{bmatrix} I_r & 0 \\ T & Z_2 \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & Z_2{}^{-1} \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ T & I_{n-r} \end{bmatrix} \in G;$$

that proves the claim. Now, we have

$$x = \begin{bmatrix} A & B \\ E & C \end{bmatrix} \in G, \quad \text{so} \quad \begin{bmatrix} I_r & 0 \\ E & I_{n-r} \end{bmatrix} \in G.$$

Let P and Q be two invertible matrices; we have

$$\begin{bmatrix} I_r & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} I_r & 0 \\ E & I_{n-r} \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I_{n-r} \end{bmatrix} = \begin{bmatrix} Q & 0 \\ PEQ & P \end{bmatrix} \in G.$$

Therefore, by the claim we obtain that

$$\begin{bmatrix} I_r & 0 \\ PEQ & I_{n-r} \end{bmatrix} \in G.$$

If we find i, j such that

$$\begin{bmatrix} I_r & 0 \\ e_{ij} & I_{n-r} \end{bmatrix} \in G;$$

then we can choose invertible matrices P and Q such that $Pe_{ij}Q = \lambda e_{lm}$ for any $1 \leqslant l \leqslant n - r$, $1 \leqslant m \leqslant r$, and $\lambda \in D^*$. Therefore for every $\lambda \in D$ and $1 \leqslant i, j \leqslant n$, $I_n + \lambda e_{ij} \in G$ so $SL_n(D) \subseteq G$ (see [10, p. 137]) and, since diagonal matrices are contained in M, we obtain $G = GL_n(D)$. Now, we find i, j such that

$$\begin{bmatrix} I_r & 0 \\ e_{ij} & I_{n-r} \end{bmatrix} \in G.$$

By the theorem of [7, p. 380] there exist invertible matrices $P$and Q such that

$$PEQ = \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}$$

for some $s \geqslant 1$. If $s = 1$ we are done. If $s \geqslant 2$ then there exists $A \in GL_{n-r}(D)$ such that $AX = X + e_{12}$ where $X = PEQ$. Now, we have

$$\begin{bmatrix} I_r & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} I_r & 0 \\ X & I_{n-r} \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ X + e_{12} & A \end{bmatrix} \in G.$$

Using the claim we conclude that

$$\begin{bmatrix} I_r & 0 \\ X + e_{12} & I_{n-r} \end{bmatrix} \in G.$$

On the other hand,

$$\begin{bmatrix} I_r & 0 \\ X + e_{12} & I_{n-r} \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{bmatrix} \begin{bmatrix} I_r & 0 \\ X & I_{n-r} \end{bmatrix} \in G.$$

Therefore

$$\begin{bmatrix} I_r & 0 \\ e_{12} & -I_{n-r} \end{bmatrix} \in G,$$

and so we find that

$$\begin{bmatrix} I_r & 0 \\ e_{12} & I_{n-r} \end{bmatrix} \in G,$$

which completes the proof. □

The following corollary completely determines the structure of reducible maximal subgroups of $GL_n(D)$, for a division ring D and a natural number n.

Corollary 1. *Let D be a division ring, n a natural number, and M a maximal subgroup of $GL_n(D)$, then M is irreducible or it is conjugate to one of the maximal subgroups given in the previous theorem.*

Lemma 8. *Let D be a division ring and G a subgroup of $GL_n(D)$ such that G is irreducible then $C_{M_n(D)}(G)$ is a division ring.*

Proof. Let $0 \neq A \in C_{M_n(D)}(G)$. It is easily checked that $\ker A$ and $\operatorname{im} A$ are two invariant subspaces of D^n under G. Now by irreducibility of G we have $\operatorname{im} A = D^n$ and $\ker A = 0$, which completes the proof. □

By the above lemma we immediately obtain the following result.

Corollary 2. *Let D be a division ring with center F and M be an abelian maximal subgroup of $GL_n(D)$ such that $n \geqslant 2$ or $D \neq F$; then $K = M \cup \{0\}$ is a field and $K = C_{M_n(D)}(M)$. Furthermore, if D is finite-dimensional over F then $[K:F]^2 = n^2[D:F]$ and there is no field between F and K.*

Proof. By Corollary 1 and Lemma 8, we have that $C_{M_n(D)}(M)$ is a division ring; so $C_{M_n(D)}(M) = K$. Now suppose that $[D : F] < \infty$. By Theorem 3 of [10, p. 45] we obtain $[K : F]^2 = n^2[D : F]$. Let $F \subsetneqq E \subsetneqq K$. We have $M < U(C_{M_n(D)}(E))$ and, so by maximality of M, we conclude $M = U(C_{M_n(D)}(E))$; so by Centralizer Theorem we have $E = C_{M_n(D)}(C_{M_n(D)}(E)) = K$, which is a contradiction. □

As a consequence of this corollary we obtain the next corollary.

Corollary 3. *If F is an algebraically closed field, then $GL_n(F)$ has no abelian maximal subgroup.*

The following corollary is a generalization of Corollary 2 of [3].

Corollary 4. *Let D be a division ring with center F with $GL_n(D)$ algebraic over F and such that $[D:F]=\infty$; then $GL_n(D)$ contains no abelian maximal subgroup.*

Proof. If $GL_n(D)$ has an abelian maximal subgroup, say M, then by Corollary 2, $M \cup \{0\}$ is a field. Let $a \in M \backslash F$. We have that $F(a)$ is a field and $m=[F(a):F]<\infty$, so by Centralizer Theorem, $M_n(D) \otimes_F F(a) \simeq C_{M_n(D)}(F(a)) \otimes_F M_m(F)$ and $C_{M_n(D)}(F(a))$ is a simple ring. Now, by maximality of M we have $U(C_{M_n(D)}(F(a)))=M$, therefore $C_{M_n(D)}(F(a))=M \cup \{0\}$. Thus $M_n(D) \otimes_F F(a) \simeq M_m(M \cup \{0\})$, which satisfies the standard polynomial identity, so $[D:F]<\infty$, a contradiction. □

Before proving the next result, we need the following interesting theorem.

Theorem E [19, p. 9]. *Let D be a division ring, F a subfield of $Z(D)$, and G a subgroup of $GL_n(D)$. Set $R=F[G]$.*

(a) *If G is completely reducible, then R is semiprime; more generally, so is $F[N]$ for every subnormal subgroup N of G.*
(b) *If G is irreducible, then R is prime.*
(c) *If G is locally finite and completely reducible, then R is semisimple.*

The following lemma is a generalization of Corollary 4 of [3].

Lemma 9. *Let D be an infinite division ring with center F and M be a maximal subgroup of $GL_n(D)$ such that $F^* \subseteq M$; then M/F^* is an infinite group.*

Proof. By contradiction, suppose that M/F^* is a finite group. At first, we prove that M is irreducible; if not, by Corollary 1 there exists a suitable matrix P and a natural number m such that $0<m<n$ and $PMP^{-1}=H$, where

$$H=\left\{\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \,\middle|\, A \in GL_m(D),\ C \in GL_{n-m}(D),\ B \in M_{m\times(n-m)}(D)\right\}.$$

On the other hand, M/F^* is a finite group, so H/F^* must be finite. For any $\lambda \in D$ let $A_\lambda = I_n + \lambda e_{1n} \in H$. It is easily seen that if $\gamma \neq \lambda$ then $A_\gamma {A_\lambda}^{-1} \notin F^* I_n$, but D is infinite, a contradiction. Thus M is irreducible and, by Theorem E, $F[M]$ is a prime ring. Now, M/F^* is a finite group, therefore $[F[M]:F]<\infty$; so $F[M]$ is an artinian ring. Thus $J(F[M])$ is a nilpotent ideal. But $F[M]$ is a prime ring, so $J(F[M])=0$, therefore $F[M]$ is a simple ring. Thus $F[M] \simeq M_k(D_1)$ for a suitable division ring D_1. If $U(F[M])=M$ then by the fact that $[M:F^*]$ is finite we conclude that $[GL_k(D_1):F^*]<\infty$; therefore D_1 is finite, that is, M is finite, which contradicts Theorem 6 of [1]. So, by maximality of M, we have $U(F[M])=GL_n(D)$; thus $F[M]=M_n(D)$, but $[F[M]:F]<\infty$, so D is a finite-dimensional division ring. Now, there exist some elements $a_1,\dots,a_r \in M$ such that $M=\bigcup_{i=1}^r F^* a_i$. Choose an element $x \in GL_n(D)\backslash M$; by maximality of M we have $\langle M,x\rangle = GL_n(D)$. Now let $H=\langle a_1,\dots,a_r,x\rangle$, so $F^*H=GL_n(D)$, therefore $H \lhd GL_n(D)$. But H is a finitely generated group and by [3,4] we conclude that H is central; thus $n=1$ and $D=F$, a contradiction. □

3. Solvable and locally nilpotent maximal subgroups of $GL_n(D)$

The maximal solvable, maximal nilpotent, and maximal locally nilpotent subgroups of general linear groups were extensively studied by Suprunenko; the main results are expounded in [20]. Our object here is to discuss the general skew linear groups whose maximal subgroups are of some special types. For instance, abelian maximal, solvable maximal, locally nilpotent maximal, torsion maximal, and locally finite maximal subgroups of general skew linear groups will be investigated. Our first result in this direction is as follows.

Lemma 10. *Let D be a division ring with at least four elements and center F. If M is a solvable maximal subgroup of $GL_n(D)$, where D is noncommutative or $n \geqslant 3$, then M is irreducible.*

Proof. By contradiction, suppose that M is reducible. So by Corollary 1 there exists a suitable matrix $P \in GL_n(D)$ and a natural number m such that $0 < m < n$ and

$$PMP^{-1} = \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \,\middle|\, A \in GL_m(D),\ C \in GL_{n-m}(D),\ B \in M_{m\times(n-m)}(D) \right\}.$$

On the other hand, M is solvable, therefore $GL_m(D)$ and $GL_{n-m}(D)$ are solvable groups. Thus D is a field and $m = n - m = 1$, so $n = 2$, which is a contradiction. $\square$

If $n = 2$, the above lemma fails. To see this, consider the solvable maximal subgroup

$$\mathrm{Tr}_2(F) = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \,\middle|\, a, c \in F^*,\ b \in F \right\}$$

of $GL_2(F)$, where F is a field. The following theorem plays an important role in our proofs.

Theorem F [19, p. 7]. *Let D be a locally finite-dimensional division ring with center F and G a subgroup of $GL_n(D)$. Set $R = F[G]$.*

(a) *If G is completely reducible, then R is semisimple.*
(b) *If G is irreducible, then R is simple.*

Lemma 11. *Let D be a locally finite-dimensional division ring over its center F with at least four elements and M a non-abelian solvable maximal subgroup of $GL_n(D)$ such that D is noncommutative or $n \geqslant 3$; then M is absolutely irreducible.*

Proof. By Lemma 10, M is irreducible. So by Theorem F, $F[M]$ is a simple ring. By maximality of M we have two cases. If $U(F[M]) = GL_n(D)$, there is nothing to prove; if not, then $U(F[M]) = M$. Now, because $F[M]$ is a simple ring and $U(F[M])$ is a solvable group, M is abelian, a contradiction. $\square$

Theorem G [19, p. 102]. *Let D be a locally finite-dimensional division ring with center F and G a locally nilpotent subgroup of $GL_n(D)$; then G is solvable.*

Corollary 5. *Let D be a division ring which is locally finite-dimensional over its center F, with at least four elements, and M a non-abelian locally nilpotent maximal subgroup of $GL_n(D)$; then M is absolutely irreducible.*

Proof. According to Lemma 11 and Theorem G, we can assume that $F = D$ and $n = 2$. Now, we claim that M is irreducible. If not, there exists a suitable matrix $P \in GL_2(F)$ such that PMP^{-1} is a subgroup of

$$\mathrm{Tr}_2(F) = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \,\middle|\, a, c \in F^*,\ b \in F \right\}.$$

By maximality of M we obtain that $\mathrm{Tr}_2(F) = PMP^{-1}$; so it is a locally nilpotent group. But choose an element $1 \neq a \in F^*$. Then the group

$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & a \end{bmatrix} \right\rangle$$

has trivial center, so it is not nilpotent, a contradiction. Therefore M is irreducible. On the other hand, using Theorem F we obtain that $F[M]$ is a simple ring. But, if $U(F[M]) = M$, then M must be an abelian group, which is a contradiction. So, by maximality of M, $U(F[M]) = GL_n(D)$ and therefore $F[M] = M_n(D)$, which completes the proof. □

Before proving the next theorem, we need the following remark which is an easy consequence of [5, p. 165].

Remark 1. Let D be a division ring with at least four elements and center F, R a semi-simple subring of $M_n(D)$, and $U(R)$ a normal subgroup of $GL_n(D)$; then R is either central or equals $M_n(D)$.

Lemma 12. *Let M be a solvable maximal subgroup of $GL_n(D)$, where D is a division ring with at least four elements, and $n \geqslant 5$. Then M is primitive.*

Proof. By contradiction suppose that M is imprimitive; then by maximality of M we obtain that $M \simeq GL_r(D) \wr S_k$, the wreath product of $GL_r(D)$ and S_k, where $rk = n$. Since M is solvable, then $GL_r(D)$ is solvable and so $r = 1$ and D is a field. Thus $k = n$ and so S_n is solvable, a contradiction. Therefore M is a primitive subgroup of $GL_n(D)$. □

Theorem 8. *Let F be an infinite field and M a non-abelian solvable maximal subgroup of $GL_n(F), n \geqslant 5$; then n is a prime number, there is a maximal subfield K of $M_n(F)$ such that $[M : K^*] = n$ and K/F is a cyclic extension, $[K : F] = n$, and the groups $\mathrm{Gal}(K/F)$ and M/K^* are isomorphic to $\mathbb{Z}_n$. Furthermore, there exists $x \in M$ such that $M = \bigcup_{i=1}^{n} K^* x^i$ and $M_n(F) = \bigoplus_{i=1}^{n} K x^i$.*

Proof. By Lemma 11, M is absolutely irreducible. Now, using Theorem 6 of [20, p. 135], M contains a maximal abelian normal subgroup, say N, such that $|M/N| < \infty$. On the other hand, by Lemma 12, M is primitive; thus using Lemma 1 of [20, p. 129] there exists a field extension E of F such that $N = E^*$. Now put $V = C_M(N)$. Since $N \lhd M$, we find that $V \lhd M$. On the other hand, according to Lemmas 2 and 3 of [20, p. 139–140], $F[V]$ is a simple ring. But we have $M < N_{GL_n(F)}(U(F[V]))$. If $F[V] = M_n(F)$, then we obtain $N \subseteq F$; so $[M : F^*]$ is finite, which contradicts Lemma 9. Thus we may assume that $F[V] \neq M_n(F)$ and, by the theorem of [5, p. 165], we have $U(F[V]) < M$. Now, since M is solvable and $F[V]$ is a simple ring, V is an abelian group and, since N is a maximal abelian normal subgroup of M, $V = N$ and $E^* = C_M(E^*)$.

Suppose that K is a maximal subfield of $M_n(F)$ containing E. Therefore we have $K^* < N_{GL_n(F)}(U(F[N]))$; hence $K^* < M$. But we know that $E^* = C_M(E^*)$ and $E \subseteq K$; thus $K = E$. Now, since M is absolutely irreducible by [20, p. 139], we have $[M_n(F) : F] = n^2/[\Sigma : F]$, where Σ/F is a field extension. So, $\Sigma = F$. Thus, by Theorem 1 of [20, p. 138], we have $M/K^* \simeq \mathrm{Gal}(K/F)$. Now, according to the argument used in the proof of Theorem 5, the proof is complete. □

Remark 2. It is interesting that by [18], the monomial subgroup of $GL_n(F)$, where F is an infinite field, is a maximal subgroup of $GL_n(F)$. Also one can check that for $n < 5$ the monomial subgroup of $GL_n(F)$ is solvable. To see this, consider the quotient group $H/D \simeq S_n$, where H is the monomial subgroup of $GL_n(F)$ and D is the diagonal subgroup of $GL_n(F)$. We also show that $GL_n(\mathbb{R})$, $n \geqslant 5$, cannot have a solvable maximal subgroup. If M is a non-abelian solvable maximal subgroup of $GL_n(\mathbb{R})$, then by Theorem 8, there exists a maximal subfield K of $M_n(\mathbb{R})$ such that M/K^* is finite and $[K : \mathbb{R}] = n$. But it is well known that every finite field extension of $\mathbb{R}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$, thus we obtain $K \simeq \mathbb{R}$ or $K \simeq \mathbb{C}$, which contradicts $n \geqslant 5$.

Now we prove that $GL_n(\mathbb{R})$, $n \geqslant 2$, has no abelian maximal subgroup. By Corollary 2, $K = M \cup \{0\}$ is a field. We know that $\mathbb{R} \subsetneqq K$, so $K \simeq \mathbb{C}$, thus $n = 2$. But $\mathrm{Gal}(K/\mathbb{R})$ is nontrivial and, by Skolem–Noether Theorem, there is an element $x \in GL_2(\mathbb{R}) \backslash K^*$ such that $xK^*x^{-1} = K^*$. We have $\langle K^*, x\rangle < N_{GL_2(\mathbb{R})}(K^*)$ and by maximality of K^* we find that $K^* \lhd GL_2(\mathbb{R})$. But every noncentral normal subgroup of $GL_2(\mathbb{R})$ contains $SL_2(\mathbb{R})$ as a subgroup, a contradiction. We even do not know if $GL_n(\mathbb{Q})$, $n > 1$, can have an abelian maximal subgroup.

It is also true that for a finite field F, $GL_n(F)$, $n \geqslant 2$, has no abelian maximal subgroup except in the case $F = \mathbb{Z}_2$ and $n = 2$. To see this we use the fact that if a finite group has an abelian maximal subgroup then it is solvable of length at most 3 [17, p. 392]. If $n \geqslant 3$ or $F \neq \mathbb{Z}_2, \mathbb{Z}_3$, we have $(SL_n(F))' = SL_n(F)$ which contradicts solvability of $GL_n(F)$. Clearly $GL_2(\mathbb{Z}_2)$ has an abelian maximal subgroup. Now we show that $G = GL_2(\mathbb{Z}_3)$ has no abelian maximal subgroup. We have $|G| = 48$. Suppose that M is an abelian maximal subgroup of G. If $M \lhd G$, then $G' < M$, which is a contradiction. So we can assume that $[G : M] > 2$. Now suppose that $[G : M] = 3$. We know that the number of 2-Sylow subgroups of G is 1 or 3. But M is not a normal subgroup of G. Hence we may assume that G has three abelian maximal subgroups, say M_1, M_2, M_3. For any $i \neq j$, we have $G = \langle M_i, M_j\rangle < C_G(M_i \cap M_j)$ and $Z(G) = M_i \cap M_j = \{\pm I\}$.

This implies that $|M_1 \cup M_2 \cup M_3| = 44$ and that G should have exactly two 3-Sylow subgroups, a contradiction. Now suppose that $[G : M] \neq 3$. Since M is not a normal subgroup of G, there exists an element $x \in G$ such that $xMx^{-1} \neq M$. We know that $Z(G) = \{\pm I\} < M$. If $a \in M \cap xMx^{-1}$, then M and xMx^{-1} are contained in $C_G(a)$. It follows that $M \cap xMx^{-1} = Z(G) = \{\pm I\}$. If $3 \nmid |M|$, then by the first Sylow's Theorem $[G : M] = 3$. So we can assume that $3 \mid |M|$, and so $6 \mid |M|$. If $|M| \neq 6$, then we have $|M| = |xMx^{-1}| \geqslant 12$, and so G has at least 72 elements, a contradiction. Hence assume that $|M| = 6$. It is well known that $[G : G'] = 2$. We know that $\{\pm I\} < M \cap G'$. Since M is maximal, we have that $M \nsubseteq G'$, so $|MG'| = 72$, which is a contradiction.

The following conjecture can be viewed as a variation of Conjecture A appeared in the introduction.

Conjecture 1. *Let D be a noncommutative division ring, then for $n > 1$, $GL_n(D)$ has no solvable maximal subgroup. Furthermore, if F is a field, then for $n \geqslant 5$, $GL_n(F)$ has no solvable maximal subgroup.*

It seems that the above conjecture remains true if one replaces "solvable" by "locally solvable". In [3] authors try to prove that if the multiplicative group of a division ring has an abelian maximal subgroup, then the division ring is a field. In what follows we want to pose a similar question for general skew linear groups.

Conjecture 2. *Let D be a division ring; then for $n > 1$, $GL_n(D)$ has no abelian maximal subgroup, and for $n = 1$, if D^* has an abelian maximal subgroup, then D is a field.*

The following lemma, apart from being useful in field theory, has a key role in the proof of Lemma 14.

Lemma 13. *Let K/F be a field extension such that K is infinite and K^*/F^* is a torsion group, then K/F is a purely inseparable extension if and only if $\Pi(K^*/F^*)$ is a finite set.*

Proof. One side is clear. To prove the other side by contradiction, suppose that K/F is not a purely inseparable extension. Since K^*/F^* is a torsion group, then by Kaplansky's Lemma, $\operatorname{Char} K = p > 0$ and K is algebraic over $\mathbb{Z}_p$. Now, for any element $x \in K \backslash F$ there exists $y \in \mathbb{Z}_p(x)$ such that $(\mathbb{Z}_p(x))^* = \langle y \rangle$ and $o(y) = |(\mathbb{Z}_p(x))^*| = p^l - 1$. But we know that $x \in K \backslash F$, so $y \in K \backslash F$. Now, let $\Pi(K^*/F^*) = \{p_1, \ldots, p_r\}$; then there exist nonnegative integers $\alpha_1, \ldots, \alpha_r$ such that $z = y^{p_1^{\alpha_1} \cdots p_r^{\alpha_r}} \in F$; thus $\mathbb{Z}_p(z)$ is a proper subfield of $\mathbb{Z}_p(x)$. Let $|\mathbb{Z}_p(z)| = p^k$, thus $k \mid l$, $k \neq l$. Therefore $y^{(p^k-1)p_1^{\alpha_1} \cdots p_r^{\alpha_r}} = 1$ and by the fact that $o(y) = p^l - 1$ we conclude that $(p^l - 1) | (p^k - 1) p_1^{\alpha_1} \ldots p_r^{\alpha_r}$; so there exist nonnegative integers $\beta_1, \ldots, \beta_r$ such that $(p^l - 1)/(p^k - 1) = p_1^{\beta_1} \ldots p_r^{\beta_r}$. On the other hand, $K \backslash F$ is an infinite set and by the fact that any nonzero polynomial has finitely many roots we can choose a suitable element x such that l is as large as we need. Now, by the theorem of [8], $p^l - 1$ has a prime factor, say q, such that for every $i \mid l$, $q \nmid p^i - 1$, and hence for any $i < l$, $q \nmid p^i - 1$ and so $q = p_j$ for some j, $1 \leqslant j \leqslant r$. Now, choose $y_1, \ldots, y_{r+1}$ such that $o(y_i) = p^{l_i} - 1$, $y_i \in K \backslash F$, and $l_1 < \cdots < l_{r+1}$. Thus, for every i, $1 \leqslant i \leqslant r + 1$, there exists a natural number j_i, $1 \leqslant j_i \leqslant r$, such that p_{j_i} is a prime factor of $p^{l_j} - 1$ and $p_{j_i} \nmid p^s - 1$, for every $s < l_j$. So p_{j_i}'s are distinct numbers, which is a contradiction. $\square$

Before proving the next result, we need the following three theorems.

Theorem H [19, p. 4]. *A subnormal subgroup of a completely reducible skew linear group is completely reducible.*

Theorem I [19, p. 215]. *Let H be a locally nilpotent normal subgroup of the absolutely irreducible subgroup G of $GL_n(D)$; then H is center by locally finite and $G/C_G(H)$ is periodic.*

Theorem J [6]. *Let G be a locally finite group and M an abelian maximal subgroup of G. Then G is solvable.*

Now we are in a position to prove the following lemma.

Lemma 14. *Let F be an infinite field that is algebraic over $\mathbb{Z}_p$, then for $n \geqslant 2$, $GL_n(F)$ has no locally nilpotent maximal subgroup.*

Proof. First we claim that if M is a locally nilpotent maximal subgroup of $GL_n(F)$, then M is an abelian group. By contradiction, suppose that M is not an abelian group. By Corollary 5 and Lemma 4, M is absolutely irreducible and $F^* \subseteq M$ and so $Z(M) = F^*$. Now, by the theorem of [20, p. 135], there exists an abelian normal subgroup N of M such that $|M/N| < \infty$. By Lemma 9, we obtain that $N \nsubseteq F^*$. On the other hand, M is completely reducible and N is a normal subgroup of M, so by Theorem H we conclude that $F[N] \simeq F_1 \times \cdots \times F_r$ (as F-algebra) for some fields F_i. But by Theorem I we obtain that M/F^* is a locally finite group. We have $N \lhd M$, so $M \subseteq N_{GL_n(F)}(U(F[N]))$ and two cases can be considered: if $N_{GL_n(F)}(U(F[N])) = GL_n(F)$, then $U(F[N]) \lhd GL_n(F)$, and, since N is a noncentral abelian subgroup, we obtain a contradiction. So $N_{GL_n(F)}(U(F[N])) = M$ and thus $U(F[N]) \subseteq M$. Now, by Theorem 1 of [20, p. 216], we obtain that $|\Pi(M/F^*)| < \infty$. We have two cases.

(i) $r = 1$. Therefore $F[N]$ is a field and $|\Pi((F[N])^*/F^*)| < \infty$, so by Lemma 13 we conclude that $F[N]/F$ is a purely inseparable extension; but F is an algebraic extension of $\mathbb{Z}_p$, which is a contradiction.

(ii) $r > 1$. We know that $U(F[N]) \subseteq M$, so $|\Pi(U(F[N])/F^*)| < \infty$. But for any $a \in F^*$ we have $b = (1, \ldots, 1, a) \in F_1^* \times \cdots \times F_r^*$; so there exists a natural number k such that all prime factors of it are in $\Pi(U(F[N])/F^*)$ and $b^k \in F^*$, so $a^k = 1$. Therefore $\Pi(F^*) \subseteq \Pi(U(F[N])/F^*)$; that is, $\Pi(F^*)$ is a finite set, and by Lemma 13 we conclude that $F/\mathbb{Z}_p$ is a purely inseparable extension, which is a contradiction. Thus the claim is proved. Now, Theorem J completes the proof. $\square$

Remark 3. We note that the above lemma is not true for $n = 1$. Let $F = \bigcup_{i=1}^{\infty} F_{2^{2^i}}$, where F_{2^r} is the finite field of order 2^r. We claim that F^* has a maximal subgroup. To see this, we note that F^* has no element of order 9, because $9 \nmid 2^{2^r} - 1$ for any natural number r; so $F^{*3} \neq F^*$. Thus F^* is not divisible and hence has a maximal subgroup.

Theorem 9. *Let D be a division ring with center F and M a locally finite maximal subgroup of $GL_n(D)$; then $D = F$, $\operatorname{Char} F = p > 0$, and F is algebraic over $\mathbb{Z}_p$.*

Proof. By the method used in the proof of Lemma 9, it is not hard to see that we can assume M is irreducible. Now, by Theorem E we obtain that $F[M]$ is a prime and semisimple ring, so it is a simple ring. On the other hand, by Lemma 2 we conclude that $F^* \subseteq M$ or $SL_n(D) \subseteq M$. But, if $SL_n(D) \subseteq M$ then $M \lhd GL_n(D)$, and so $[GL_n(D) : M] < \infty$. Thus we conclude that $GL_n(D)$ is a torsion group; so $D = F$ and F is algebraic over $\mathbb{Z}_p$, a contradiction. Hence we may assume that $F^* \subseteq M$. Therefore $\operatorname{Char} F = p > 0$ and F is algebraic over $\mathbb{Z}_p$.

Now, by the fact that $F[M]$ is a simple ring, one concludes that $Z(F[M])$ is a field. In fact, we claim that $F^* = Z(M)$. Let $x \in Z(M)\setminus F$. By the fact that x is torsion and $F[x]$ is a field, we can use Skolem–Noether Theorem and find $y \in GL_n(D)$ such that $yxy^{-1} = x^i \neq x$. Therefore $\langle y, M\rangle \subseteq N_{GL_n(D)}(\langle x\rangle)$. Thus, maximality of M yields that $\langle x\rangle \lhd GL_n(D)$, which is a contradiction, because if $n = 1$, then $(F[x])^*$ is a normal subgroup of D^* which contradicts Cartan–Brauer–Hua Theorem, and if $n > 1$, then $SL_n(D)$ must be a subgroup of $\langle x\rangle$, and it is obviously a contradiction. So, $F^* = Z(M)$.

Now, we have two cases. If $U(F[M]) = GL_n(D)$, then $F[M] = M_n(D)$. So for any $a \in GL_n(D)$, there exist m_i's in M such that, $a = m_1 + \cdots + m_r$. Now, since M is a locally finite group, $\mathbb{Z}_p[\langle m_1, \ldots, m_r\rangle]$ is a finite ring. Thus a must be torsion, which completes the proof. In the other case, we have $U(F[M]) = M$. Since $F[M]$ is a simple ring, there exists a division ring D_1 such that, $F[M] \simeq M_m(D_1)$ as F-algebra and, since $F^* = Z(M)$, we have $F = Z(D_1)$. Now, by the fact that M is torsion and Jacobson's Theorem [13, p. 219], we obtain $D_1 = F$. So $F[M]$ is a finite-dimensional simple ring with center F. By Centralizer Theorem, we have that

$$M_n(D) \otimes_F (F[M])^{\mathrm{op}} \simeq C_{M_n(D)}(F[M]) \otimes_F M_{m^2}(F) \qquad (*)$$

and $C_{M_n(D)}(F[M])$ is a simple ring. Since M is a maximal subgroup which is not normal in $GL_n(D)$, $N_{GL_n(D)}(M) = M$. So $U(C_{M_n(D)}(F[M])) \subseteq M$, therefore $[C_{M_n(D)}(F[M]) : F] < \infty$. Now, by $(*)$ we can conclude that D is a finite-dimensional division ring over F. But F is algebraic over $\mathbb{Z}_p$, so again by Jacobson's Theorem $D = F$, which completes the proof. □

According to previous result, we pose the following conjecture.

Conjecture 3. *Let D be a division ring and M a torsion maximal subgroup of $GL_n(D)$; then D is a field and is algebraic over $\mathbb{Z}_p$.*

Theorem K [19, p. 14]. *Let G be an absolutely irreducible subgroup of $GL_n(D)$, N a normal subgroup of G, and K a subring of $M_n(D)$ normalized by G. If $Z(D) \subseteq K$ and G/N is locally finite, then $K[N]$ is semisimple.*

Maximal locally nilpotent linear groups has been described in details by D.A. Suprunenko [20]. The following theorem shows that locally nilpotent maximal subgroups of $GL_n(D)$, where D is a finite-dimensional division ring, should be abelian.

Theorem 10. *Let D be a finite-dimensional division ring with center F, which is infinite, and M a locally nilpotent maximal subgroup of $GL_n(D)$; then M is an abelian group.*

Proof. By contradiction, suppose that M is non-abelian. So by Corollary 5 and Lemma 4, M is absolutely irreducible and $F^* \subseteq M$, so $Z(M) = F^*$. Since D is a finite-dimensional division ring, we can assume $GL_n(D)$ as a subgroup of $GL_k(F)$ for some suitable k. By Theorem I we conclude that M/F^* is a locally finite group; so by Lemma 3, M' is locally finite. We divide the proof into three steps.

Step 1 ($M' \nsubseteq F$). By contradiction, suppose $M' < F^*$. Obviously, for any $a \in M\backslash F$ we have that $F^*\langle a\rangle \lhd M$ and, in particular, that $M/F^*\langle a\rangle$ is locally finite. On the other hand, M is absolutely irreducible, so, by Theorem K, $F[a]$ is a semisimple ring. Thus $F[a] \simeq F_1 \times \cdots \times F_r$ (as F-algebra) for suitable fields, F_i's. But $F[a]$ is a noncentral and commutative subring of $M_n(D)$, and M is a subgroup of $N_{GL_n(D)}(U(F[a]))$; so by Remark 1 it is concluded that $U(F[a]) < M$. Now, we prove that $F[a]$ is a field. If not, let $b = (1, \ldots, 1, x)$ for an arbitrary $x \in F^*$. Now, since M/F^* is a torsion group, there exists a natural number m such that $b^m \in F^*$, and therefore $x^m = 1$. So F is algebraic over $\mathbb{Z}_p$ and, by Jacobson's Theorem, $D = F$, which contradicts Lemma 14. So we obtain that $F[a]$ is a field, in particular, $a + 1 \in M$. But $a \notin F^*$, so there exists $c \in M\backslash C_M(a)$. On the other hand, we have:

$$cac^{-1}a^{-1} - 1 = \bigl(cac^{-1}a^{-1} - c(a+1)c^{-1}(a+1)^{-1}\bigr)(a+1).$$

Therefore, $a + 1$ must be an element of F, a contradiction.

Step 2 (M' is finite). First we claim that if N is a normal subgroup of M such that $N < M'$ and there exists an element $z \in N\backslash F$ such that $[N : C_N(z)] < \infty$, then N is a finite group.

By finiteness of $[N : C_N(z)]$, there exists a subgroup K of $C_N(z)$ such that $K \lhd N$ and $[N : K] < \infty$. Let $m = |N/K|$ and $G = \langle x^m,\ x \in N\rangle$. We prove that $G \subseteq F$. Suppose $G \nsubseteq F$. By the fact that N is a normal subgroup of M, we conclude that $G \lhd M$; therefore $M \subseteq N_{GL_n(D)}(U(F[G]))$. Now, since M is completely reducible and $G \lhd M$, by Theorem H we obtain that G is also completely reducible. On the other hand, G is locally finite, so by Theorem E it is concluded that $F[G]$ is a semisimple ring. Now, by maximality of M we have two cases. If $N_{GL_n(D)}(U(F[G])) = GL_n(D)$ then $U(F[G]) \lhd GL_n(D)$. But $G \nsubseteq F$, so by Remark 1 we have $F[G] = M_n(D)$; but $z \in C_{GL_n(D)}(G)\backslash F$, which is a contradiction. Therefore $N_{GL_n(D)}(U(F[G])) = M$. So $U(F[G])$ is locally solvable and, by Zassenhaus' Theorem [20, p. 137], it is a solvable group. Now, by the fact that $F[G]$ is a semisimple ring, we conclude that there exist suitable field extensions of F, F_i's, such that $F[G] \simeq F_1 \times \cdots \times F_r$ (as F-algebra). As we noted in the first step, we have $r = 1$. So $F[G]$ is a field and $(F[G])^*/F^*$ is a torsion group. Now, by Kaplansky's Lemma and Lemma 14, $F[G]/F$ must be a purely inseparable extension. But we know that G is a torsion group, so $G \subseteq F^*$. Let $x \in N$; so $x^m \in G \subseteq F^*$, therefore $\det(x^m) = x^{mk}$, where $k = n[D : F]$. But $N \subseteq M'$, thus $\det(x) = 1$. So it is concluded that $x^{mk} = 1$. On the other hand, N is a normal subgroup of M, and M is completely reducible, so by Theorem H, N is also completely reducible. Now, by Burnside's Theorem we conclude that N is a finite group, which proves the claim.

Now, let r be the largest number such that $M^r \nsubseteq F^*$. Thus for any $x \in M^{r+1}$ we have $x^k = \det(x) = 1$. Therefore, since $M^{r+1} \lhd M$, M^{r+1} is completely reducible and so M^{r+1} is finite, which implies that M^r is an FC group. Now, by the above claim and Step 1, we obtain that M^r is finite. By continuing this method, we find that M' is finite.

Step 3. By finiteness of M', we obtain that M is an FC group and, on the other hand, it is a linear group; so by Theorem 4 of [20, p. 180] we conclude that M/F^* is a finite group, which contradicts Lemma 9. □

In [3] it is shown that if D is a division ring with center F and M is a maximal subgroup of D containing F^*, then M/F^* cannot be a finite group. In relation to $GL_n(D)$, we have following result.

Theorem 11. *Let D be a finite-dimensional division ring with center F and* $\operatorname{Char} F = 0$, *$n$ a natural number, and M a maximal subgroup of $GL_n(D)$ such that $F^* \subseteq M$; then M/F^* cannot be a locally finite group.*

Proof. By contradiction, suppose that M/F^* is locally finite. If M is reducible, then by Corollary 1 we find that M is conjugate to one of the maximal subgroups given in Theorem 7. So, there exists a natural number r such that $(I_n + e_{1n})^r \in F$, which contradicts $\operatorname{Char} F = 0$. Thus M is irreducible. It implies that $F[M]$ is a simple ring, hence $F[M] \simeq M_s(D_1)$, where D_1 is a division ring. Now maximality of M yields that M is absolutely irreducible. Since otherwise $GL_s(D_1) = U(F[M]) = M$, thus $GL_s(D_1)/F^*$ is a torsion group and, by Kaplansky's Lemma and the fact that $\operatorname{Char} F = 0$, we obtain that $D_1 = F$ and $s = 1$. So M is abelian and, by Corollary 2, we have that $M \cup \{0\}$ is a field but M/F^* is torsion; so Kaplansky's Lemma implies that $\operatorname{Char} F \neq 0$, which is a contradiction. Thus M is absolutely irreducible, and $Z(M) = F^*$. Now, we claim that if $N \lhd M$ then $F[N] = M_n(D)$ or $N \subseteq F$. We know that M is completely reducible, so $F[N]$ is semisimple. On the other hand, $M \subseteq N_{GL_n(D)}(U(F[N]))$. If $U(F[N]) \lhd GL_n(D)$ then by Remark 1 we obtain that $N \subseteq F$ or $F[N] = M_n(D)$. If $U(F[N]) \subseteq M$, then by the fact that $F[N]$ is semisimple and $U(F[N])/F^*$ is locally finite and using Kaplansky's Lemma we conclude that $F[N] = F$ or $\operatorname{Char} F = p$. This completes proof of the claim. Now, by Theorem 10, M is not nilpotent; so $M' \nsubseteq F$, thus $F[M'] = M_n(D)$. Since M/F^* is a locally finite group, then by Lemma 3, M' is locally finite. On the other hand, $\operatorname{Char} F = 0$, so M' is nonmodular and, by a theorem of [9, p. 161], it contains a normal abelian subgroup of finite index, say N. Let $[M' : N] = m$ and $G = \langle x^m,\ x \in M' \rangle$ which is a normal subgroup of M. Two cases may occur. If $G \subseteq F$ then for every $x \in M'$ we have $x^m \in F$; thus $\det(x^m) = x^{mk} = 1$ where $k = n[D : F]$ and, by Burnside's Theorem, we conclude that

M' is finite, so M is an FC group; therefore M/F^* is finite which contradicts Lemma 9. If $F[G] = M_n(D)$, then $M_n(D)$ is commutative, a contradiction. $\square$

If in Theorem 10 we omit finite dimensionality of D and add absolutely irreducibility to M, then we obtain the following result.

Theorem 12. *Let D be an infinite division ring with center F and M a locally nilpotent maximal subgroup of $GL_n(D)$ which is absolutely irreducible*; *then $D = F$ and $n = 1$*; *in particular, M is abelian.*

Proof. Since M is locally nilpotent and absolutely irreducible, by Theorem I, M/F^* is locally finite. So by Lemma 3, M' is locally finite. Now, we divide the proof into two steps.

Step 1. Suppose that $M' \subseteq F^*$. Thus we obtain that, for any $y \in M \backslash F$, $U(F[y])$ is a normal subgroup of M. Therefore, by Theorem H, $U(F[y])$ is completely reducible and, by Theorem E, $F[y]$ is semiprime. But y is algebraic over F, so $F[y]$ is an artinian ring. Thus it is a semisimple ring. So there exist suitable fields F_i's such that $F[y] \simeq F_1 \times \cdots \times F_k$. Now, by $U(F[y]) \lhd M$ and the fact that any polynomial with coefficients in F has finitely many roots in its extensions, it is concluded that y has finitely many conjugates in M. So $C_M(y)$ is finite index in M. Now, since $M/C_M(y)$ is finite, M is absolutely irreducible. Furthermore, since $U(F[y]) \lhd M$, we conclude that $C_M(y) \lhd M$ and, by Theorem K, $F[C_M(y)]$ is semisimple. On the other hand, $M \subseteq N_{GL_n(D)}(U(F[C_M(y)]))$. So, by the fact that $y \notin F$ and Remark 1, we obtain $U(F[C_M(y)]) \subseteq M$. We know that M is locally nilpotent and $F[C_M(y)]$ is semisimple, so there exist suitable fields E_i's such that $F[C_M(y)] \simeq E_1 \times \cdots \times E_t$. Now, we claim that $t = 1$. If not, for any $a \in F^*$, by the facts that M/F^* is torsion and $U(F[C_M(y)]) \subseteq M$, we can find a natural number m such that $(1, \ldots, 1, a^m) \in F^*$, which implies that $F/\mathbb{Z}_p$ is algebraic. Therefore M/F^* and F^* are locally finite and so M is also locally finite, which by Theorem 9 and Lemma 14 means that we are done. So $F[C_M(y)]$ is a field. Now, by the fact that $C_M(y)$ is finite index in M, we obtain that $F[M] = M_n(D)$ is of finite dimension over $F[C_M(y)]$. Thus, by Lemma 6, D is finite-dimensional over F and Theorem 10 completes the proof.

Let r be a natural number such that $M^r \nsubseteq F$. Now, we claim that $F[M^r] = M_n(D)$. Since M/F^*M^r is locally finite and M is absolutely irreducible, by Theorem K we conclude that $F[M^r]$ is a semisimple ring. On the other hand, $M \subseteq N_{GL_n(D)}(U(F[M^r]))$, so if $F[M^r] \neq M_n(D)$, then it is concluded that $U(F[M^r]) \subseteq M$. Thus, by the fact that M is locally nilpotent, it is concluded that there exist suitable fields, F_i's, such that $F[M^r] \simeq F_1 \times \cdots \times F_k$. With the same argument as we did before, $k = 1$. So $F[M^r]$ is a field. Thus by the fact that M/F^* is locally finite, we obtain that $F[M^r]/F$ is radical and so by Kaplansky's Lemma we find that F is algebraic over $\mathbb{Z}_p$ or that $F[M^r]/F$ is a purely inseparable extension. But as we noted before, F cannot be algebraic over $\mathbb{Z}_p$. So $F[M^r]/F$ is a purely inseparable extension. Now, since M^r is torsion and $F[M^r]/F$ is a purely inseparable extension, we conclude that $M^r \subseteq F$, a contradiction. Therefore $F[M^r] = M_n(D)$ and so $Z(M^r) \subseteq F$.

Step 2. Assume that $M' \nsubseteq F$. Thus $F[M'] = M_n(D)$. We show that $M_n(D)$ is locally finite-dimensional over F. To see this, let $x_1, \dots, x_s \in M'$; then $H = \langle x_1, \dots, x_s \rangle$ is finite, thus $[F[H] : F] < \infty$. Now, by Theorem G, M is solvable. Let r be the largest natural number such that $M^r \nsubseteq F$. Since $M^{r+1} \subseteq F$, we conclude that for any $y \in M^r \backslash F$, $Z(M^r)\langle y \rangle \lhd M^r$. So, by Theorem K, $F[y]$ is a semisimple ring. Now, with the same argument as in the first step, we can conclude that y has finitely many conjugates in M^r. So, $m = [M^r : C_{M^r}(y)] < \infty$, and obviously $C_{M^r}(y) \lhd M^r$. Let $G = \langle x^m, x \in M^r \rangle$. By the fact that $M^r \lhd M$, we conclude that $G \lhd M$, and so $M \subseteq N_{GL_n(D)}(U(F[G]))$. But G is locally finite and completely reducible, so by Theorem E, $F[G]$ is semisimple. On the other hand, by maximality of M, we obtain that $U(F[G]) \lhd GL_n(D)$ or $U(F[G]) \subseteq M$. In the first case, by Remark 1 and the fact that $y \notin F$, it is concluded that $G \subseteq F$. In the other case, by the fact that M is locally nilpotent and $F[G]$ is a semisimple ring, we conclude that there exist suitable fields, E_i's, such that $F[G] \simeq E_1 \times \cdots \times E_t$. With the same argument as we did before, $F[G]$ is a field. So $F[G]/F$ is radical. Now, by Kaplansky's Lemma and the same argument of Step 1, we conclude that $F[G]$ is a purely inseparable extension of F. Since G is torsion and $F[G]/F$ is a purely inseparable extension, we can conclude $G \subseteq F$. So, in any cases we conclude $G \subseteq F$. Thus for any $x \in M^r$ we have $x^m \in F$. Now, for any $x, y \in M^r$ we have $xyx^{-1}y^{-1} = f \in F$, thus $xyx^{-1} = fy$. Therefore $f^m = 1$, and we conclude that M^{r+1} is finite. So, M^r is an FC group.

If $\operatorname{Char} D = 0$, then by the fact that M^r is locally finite and Theorem D, we conclude that M^r can be considered as a linear group. If $\operatorname{Char} D = p > 0$, since M^r is completely reducible, then by Theorem E we conclude that $\mathbb{Z}_p[M^r]$ is semisimple and prime, so it is simple, but M^r is locally finite, so $U(\mathbb{Z}_p[M^r])$ is torsion. Therefore $\mathbb{Z}_p[M^r] \simeq M_{n_1}(F_1)$ for a field F_1 which is algebraic over $\mathbb{Z}_p$. But $F[M^r] = M_n(D)$, thus $F_1 \subseteq F$, and so we conclude that $[D : F] < \infty$. Thus M^r is a linear group and hence $M^r/Z(M^r)$ is finite. So $[F[M^r] : F] < \infty$, and Theorem 10 completes the proof. □

Corollary 6. *Let D be a division ring with center F and M a nilpotent maximal subgroup of D^*. If $F[M]\backslash F$ contains an algebraic element over F, then M is an abelian group.*

Proof. Let $x \in F[M]\backslash F$ be an algebraic element over F. Clearly, we have $x^{-1} \in F[M]$. If $U(F[M]) = M$, then we have $x \in M$ and, by Theorem 4, the proof is complete. Thus we may assume that $U(F[M]) \neq M$, and so $F[M] = D$. Now by Theorem 12 the proof is complete. □

Combining Theorem 12 with Corollary 5, yields the following theorem.

Theorem 13. *Let D be a locally finite-dimensional division ring with center F, which is infinite, and M a locally nilpotent maximal subgroup of $GL_n(D)$; then M is an abelian group.*

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